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F-theory from Dirichlet 3-branes

Dileep P. Jatkar and S. Kalyana Rama

Mehta Research Institute of Mathematics & Mathematical Physics
10, Kasturba Gandhi Marg, Allahabad 211 002, India.

email: dileep, krama@mri.ernet.in

Abstract

Starting from the type IIB Dirichlet 3-brane action, we obtain a Nambu-Goto action. It is interpreted as the world volume action of a fundamental 3-brane, and its target space theory as F-theory. The target space is twelve dimensional, with signature $(11, 1)$. It is an elliptic fibration over a ten dimensional base space. The $SL(2, Z)$ symmetry of type IIB string has now an explicit geometric interpretation. Also, one gets a glimpse of the conjectured self-duality of M-theory.

1 Introduction

The strong-weak coupling duality in the string theory [1]-[3] has shed a lot of light on the non-perturbative physics of string theory. In particular, different string theories are related to each other by these duality transformations [4]-[6]. It was also realised that the RR sector of type II strings contains a wealth of information about the nonperturbative aspects of type II strings. However, the solutions carrying RR charges do not appear in the elementary excitation spectrum and, hence, there was no satisfactory understanding of the RR charged objects. But, recently, in a beautiful paper, Polchinski [7] has shown that these solitons with RR charges are precisely the Dirichlet branes, much studied earlier on [8]. From the burst of activity that followed this discovery, one now has a deeper understanding of the duality symmetries of various strings, and the relations among them [9]-[11].

One ramification of these developments is the emergence of an unifying eleven dimensional theory, referred to as M-theory, which describes the target space theory of fundamental 2-branes. All known string theories are obtained from M-theory after appropriate dimensional reductions. The geometric origin of many of the duality symmetries is also clear in M-theory [5, 6, 10]. One exception is type IIB strings, whose connection to M-theory and to other strings is understood only after compactifying these theories to nine, or lower, dimensions [6]. Also unknown is the geometric origin of type IIB duality symmetries.

Now, from the works of [12]-[22], there appears to exist a twelve dimensional theory, dubbed F-theory, which is likely to describe the target space theory of fundamental 3-branes. F-theory is only now beginning to be explored, but it already holds many promises. It appears to incorporate type IIB strings naturally and provides new insights upon compactification to lower dimensions. Quite excitingly, its compactification to four dimensions can provide a realisation, as pointed out in [13], of Witten's novel proposal towards solving the cosmological constant problem [23].

Returning to M-theory, the appearance of the eleven dimensional spacetime can also be seen from a different point of view. Type II strings have Dirichlet p-branes, with p even (odd) for type IIA (B) [8, 24, 25]. Starting from the Dirichlet 2-brane action [8, 26, 27], a Nambu-Goto action for a fundamental 2-brane has been obtained in [27, 15] (also see [28]). The target space of the fundamental 2-brane turns out to be eleven dimensional, which is the M-theory spacetime. For $p = 1$, this method gives the ten dimensional target space of the fundamental string [27].

Therefore, one may naturally hope that starting from the Dirichlet 3-brane action, a Nambu-Goto action can be obtained which can be interpreted as the world volume action of a fundamental 3-brane, and its target space theory as F-theory. The F-theory thus derived is likely to offer insights into the duality symmetries of type IIB strings. 3-branes are also special for another reason. Any p-brane of type II strings can be mapped, by Poincare duality, to a $(6-p)$-brane. Thus, $(p>3)$-branes can be equivalently described by $(p<3)$-branes. However, 3-branes are mapped onto 3-branes themselves and, hence, they must be considered on their own right.

Dirichlet 3-branes have, in fact, been considered in [15]. However, it was found that applying the techniques of [27, 15] to the Dirichlet 3-brane action gives back the original action only, in terms of a dual gauge field. It is not a Nambu-Goto action, and has no interpretation as the theory of a fundamental 3-brane and, hence, does not lead to F-theory.

In this paper, we study the 3-branes. We start from the type IIB Dirichlet 3-brane action and adopt a simple but direct approach, in some ways analogous to that of [27, 15]. Namely, we first perform a double dimensional reduction and then apply the methods of [27, 15] to the resulting action. We find the following:

The action thus obtained is a Nambu-Goto action, and can be interpreted as the world volume action of a fundamental 3-brane. The target space is twelve dimensional, with signature $(11,1)$. It is locally a product of a ten dimensional spacetime and a torus, with a fixed Kähler class. That is, the twelve dimensional target space is an elliptic fibration, having the structure of a fiber bundle whose fiber is a two dimensional torus over a ten dimensional base space [13].

The $SL(2,Z)$ symmetry of type IIB string has an explicit geometric interpretation: the $SL(2,Z)$ modulus of type IIB string is the complex structure modulus of the torus fiber.[1] The strong-weak coupling duality of type IIB exchanges the coordinates of the torus. Also, the two radii of the torus are proportional to $e^{-\phi}$ and e^{ϕ}, where $e^{-\phi}$ is the string coupling. Thus, both in the strong and weak coupling limit, the twelve dimensional theory appears to be eleven dimensional with (10,1) signature. Perhaps, this is related to the conjectured self-duality of M-theory [29].

Because of the double dimensional reduction, the Nambu-Goto action for the 3-brane appears in a gauge fixed form, where two conditions need to be imposed. However, this

[1] That such a geometric interpretation of the $SL(2,Z)$ symmetry of type IIB theory exists was anticipated in [6, 13].

is a generic phenomenon, as will be shown by a simple counting argument. That is, two such conditions are always necessary if a 3-brane Nambu-Goto action is derivable from the Born-Infeld action of a Dirichlet 3-brane. The resulting target space action does not have the full twelve dimensional general coordinate invariance. Such a possibility has been mentioned in [15], as perhaps a way of circumventing Nahm's theorem regarding realisations of supersymmetry in spacetime with dimensions > 11 [30].

The paper is organised as follows. In section 2, we start from the Born-Infeld action of a type IIB Dirichlet 3-brane, and derive the Nambu-Goto action for a fundamental 3-brane. In section 3, we obtain the target space metric of the fundamental 3-brane, in a form where the duality symmetries become explicit. In section 4, we describe the origin of the duality symmetry of IIB strings. In section 5, we conclude with a brief discussion.

2 Nambu-Goto Action

In the following, we study the Dirichlet (D-) 3-brane of the ten dimensional type IIB string. In our notation, X^μ denotes the coordinates of the ten dimensional target space of type IIB string and ξ^α, those of the 3-brane world volume. The massless fields of the type IIB string are the following: a dilaton ϕ, a metric tensor $g_{\mu\nu}$, and a second rank antisymmetric tensor $B_{\mu\nu}$ in Neveu-Schwarz (NS) sector; χ, the antisymmetric tensors $C_{\mu\nu}$, and $A_{\mu\nu\rho\sigma}$ of rank 2 and 4 respectively in RR sector. These target space tensor fields induce $g_{\alpha\beta},\ B_{\alpha\beta},\ C_{\alpha\beta}$, and $A_{\alpha\beta\gamma\delta}$ on the D-3-brane. For example,

$$g_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu} \ . \tag{1}$$

The RR charges are carried by D-instanton, D-string and D-3-brane. In the presence of the D-branes, there will also be a boundary gauge field A_α in the NS sector. Also, define

$$\mathcal{F}_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + \frac{1}{2}(B_{\alpha\beta} - B_{\beta\alpha}) \ . \tag{2}$$

The action for the D-3-brane of the type IIB string can be written as [27, 15]

$$S_{BI} = \int d^4\xi \left(e^{-\phi}\sqrt{-gP} + Q \right) \ , \tag{3}$$

where $g = \det(g_{\alpha\beta}),\ \alpha, \beta, \ldots \in \{0, 1, 2, 3\}$,

$$\begin{aligned} P &= \det(\mathbf{1} + \mathcal{F}_\alpha^\beta) = 1 + \frac{1}{2}\mathcal{F}_{\alpha\beta}\mathcal{F}^{\alpha\beta} - \frac{1}{64}(\epsilon^{\alpha\beta\gamma\delta}\mathcal{F}_{\alpha\beta}\mathcal{F}_{\gamma\delta})^2 \\ Q &= \frac{\epsilon^{\alpha\beta\gamma\delta}}{24}(A_{\alpha\beta\gamma\delta} + 6C_{\alpha\beta}\mathcal{F}_{\gamma\delta} + 3\chi\mathcal{F}_{\alpha\beta}\mathcal{F}_{\gamma\delta}) \ , \end{aligned} \tag{4}$$

and $g^{\alpha\beta}$ is used to raise the indices.

Note that the term $e^{-\phi}\sqrt{-gP}$ in (3) can be written as

$$\sqrt{-\det(e^{-\frac{\phi}{2}}g_{\alpha\beta} + e^{-\frac{\phi}{2}}\mathcal{F}_{\alpha\beta})}\ . \tag{5}$$

Now, if one makes the replacements

$$\begin{aligned} g_{\alpha\beta} &\rightarrow e^{-\frac{\phi}{2}}g_{\alpha\beta} \\ \mathcal{F}_{\alpha\beta} &\rightarrow e^{-\frac{\phi}{2}}\mathcal{F}_{\alpha\beta} \\ C_{\alpha\beta} &\rightarrow e^{\frac{\phi}{2}}C_{\alpha\beta} \\ \chi &\rightarrow e^{\phi}\chi\ , \end{aligned} \tag{6}$$

then the resulting action, in terms of the replaced variables, is the same as that in (3) but with $\phi = 0$. Therefore, we set $\phi = 0$ in this section, and restore its dependence in the next section.

At this stage, following [27, 15], one may introduce a Lagrange multiplier enforcing equation (2) as a constraint and, then, eliminate both A_α and $\mathcal{F}_{\alpha\beta}$, now treated as independent fields. For D-2-brane, the resulting action is the Nambu-Goto action of a fundamental 2-brane, with eleven dimensional target space [27, 15]. The dual of A_α is the extra world volume scalar, which becomes the eleventh target space coordinate.

For 3-brane, this procedure does not lead to the Nambu-Goto action, but leads back to the action (3), now in terms of a dual gauge field [15]. [2] But, this action is not in the Nambu-Goto form and, hence, cannot be interpreted as a world volume action of a fundamental 3-brane.

Therefore, we proceed differently at this stage. We first perform a double dimensional reduction, by setting $\xi^3 = X^9$ and taking the fields to be independent of this coordinate. We also define, for the sake of convenience, the following:

$$\begin{aligned} &A_3 = U\ ,\ \ b_m = \partial_m U + B_m\ ,\ A_{lmn} = A_{lmn3}\ , \\ &B_m = \frac{1}{2}(B_{m3} - B_{3m})\ ,\ \ C_m = \frac{1}{2}(C_{m3} - C_{3m})\ , \end{aligned} \tag{7}$$

where $l, m, \ldots \in \{0, 1, 2\}$. Then, after a straight forward algebra, various terms in (4) become:

$$\mathcal{F}_{\alpha\beta}\mathcal{F}^{\alpha\beta} \;=\; \mathcal{F}_{mn}\mathcal{F}^{mn} + 2b_m b^m$$

[2]Actually, this is so only after the addition of a term $\propto \epsilon^{\alpha\beta\gamma\delta}B_{\alpha\beta}C_{\gamma\delta}$ which, however, spoils the invariance of (3) under $\delta B_{\alpha\beta} = \partial_\alpha\Lambda_\beta - \partial_\beta\Lambda_\alpha,\ \ \delta A_\alpha = -\Lambda_\alpha$. To restore this invariance, one needs to postulate an anamolous transformation law for $A_{\alpha\beta\gamma\delta}$, of the type described in [31].

$$
\begin{aligned}
\epsilon^{\alpha\beta\gamma\delta}\mathcal{F}_{\alpha\beta}\mathcal{F}_{\gamma\delta} &= 4\epsilon^{lmn}b_l\mathcal{F}_{mn} \\
\epsilon^{\alpha\beta\gamma\delta}A_{\alpha\beta\gamma\delta} &= 4\epsilon^{lmn}A_{lmn} \\
\epsilon^{\alpha\beta\gamma\delta}C_{\alpha\beta}\mathcal{F}_{\gamma\delta} &= 2\epsilon^{lmn}(C_l\mathcal{F}_{mn} + b_lC_{mn}) \ .
\end{aligned} \tag{8}
$$

where, now, g^{mn} is used to raise the indices and

$$
\mathcal{F}_{mn} = \partial_m A_n - \partial_n A_m + \frac{1}{2}(B_{mn} - B_{nm}) \ . \tag{9}
$$

Then, P and Q in (4) become

$$
\begin{aligned}
P &= 1 + b_m b^m + \frac{1}{2}\mathcal{F}_{mn}\mathcal{F}^{mn} - \frac{1}{4}(\epsilon^{lmn}b_l\mathcal{F}_{mn})^2 \\
Q &= \frac{\epsilon^{lmn}}{6}\left(A_{lmn} + 3(C_l + \chi b_l)\mathcal{F}_{mn} + 3b_lC_{mn}\right) \ .
\end{aligned} \tag{10}
$$

Now, as in [27, 15], we introduce a Lagrange multiplier $\Lambda^{mn}(= -\Lambda^{nm})$ enforcing equation (9) as a constraint and, then, eliminate both A_m and $\mathcal{F}_{mn}$, now treated as independent fields. The total action, including the Lagrange multiplier term, is given by

$$
S = L\int d^3\xi\left(\sqrt{-gP} + Q - \frac{\Lambda^{mn}}{2}(\mathcal{F}_{mn} - \partial_m A_n + \partial_n A_m - \frac{1}{2}(B_{mn} + B_{nm}))\right) \ . \tag{11}
$$

where P and Q are given by (10), and $L = \int d\xi^3$ is the length of the ξ^3 coordinate. (Note that the integrand is independent of ξ^3 and, hence, $d\xi^3$ integral is trivial to perform.) Note that when restoring ϕ-dependence, one should also make the replacement

$$
\Lambda^{mn} \to e^{\frac{\phi}{2}}\Lambda^{mn} \ , \tag{12}
$$

along with the others given in (6). Eliminating A_m from (11) now gives $\partial_m\Lambda^{mn} = 0$, which is solved identically by

$$
\Lambda^{mn} = \epsilon^{lmn}\partial_l V \ , \tag{13}
$$

where V is a scalar, dual to A_m.

Collecting the terms linear in $\mathcal{F}_{mn}$ and defining $\lambda_m = \partial_m V - C_m - \chi b_m$, and

$$
\hat{A}_{lmn} = A_{lmn} + 3b_{[l}C_{mn]} + 6\partial_{[l}V B_{mn]}
$$

where [..] denotes total antisymmetrisation with respect to the enclosed indices, we can write the action (11) as

$$
S = L\int d^3\xi\left(e^{-\phi}\sqrt{-gP} - \frac{\epsilon^{lmn}}{2}\lambda_l\mathcal{F}_{mn} + \frac{\epsilon^{lmn}}{6}\hat{A}_{lmn}\right) \ . \tag{14}
$$

Now, it is straightforward to eliminate the field $\mathcal{F}_{mn}$ also. Varying the action (14) with respect to $\mathcal{F}_{mn}$ gives

$$\sqrt{-g}(\mathcal{F}^{mn} - \frac{1}{2}\epsilon^{lmn}\epsilon^{pqr}b_l b_p \mathcal{F}_{qr}) = -2\epsilon^{lmn}\lambda_l\sqrt{P} \ .$$

Multiplying this equation once by $\mathcal{F}_{mn}$, once by $\epsilon_{kmn}b^k$, and squaring each side of the above expression, result in three different equations. After some manipulations of these equations, $\mathcal{F}_{mn}$ can be completely eliminated from the action in (14). The action (14) can, finally, be written as

$$S = L\int d^3\xi \left(\sqrt{-g\mathcal{P}} + \frac{\epsilon^{lmn}}{6}\hat{A}_{lmn}\right) \ , \tag{15}$$

where

$$\mathcal{P} = 1 + b_m b^m + \lambda_m\lambda^m + (b_m b^m)(\lambda_n\lambda^n) - (b_m\lambda^m)^2$$

is nothing but a determinant ! That is,

$$\mathcal{P} = \det(\mathbf{1} + b_m b^n + \lambda_m\lambda^n) \ ,$$

which can be seen simply by evaluating the determinant. The above expression can equivalently be written as

$$\mathcal{P} = \det(\mathbf{1} + b_\alpha b^\beta + \lambda_\alpha\lambda^\beta) \ , \tag{16}$$

with $\alpha, \beta = 0, 1, 2, 3$, and $b_3 = \lambda_3 = 0$.

Hence, the action (14), or equivalently (15), which is the dimensionally reduced type IIB Dirichlet 3-brane action, can be written as

$$S = \int d^4\xi \left(\sqrt{-\det(g_{\alpha\beta} + b_\alpha b_\beta + \lambda_\alpha\lambda_\beta)} + \frac{\epsilon^{\alpha\beta\gamma\delta}}{6}\hat{A}_{\alpha\beta\gamma\delta}\right) \ , \tag{17}$$

where

$$\begin{aligned} b_\alpha &= \partial_\alpha U + B_\alpha \ , \\ \lambda_\alpha &= \partial_\alpha V - C_\alpha - \chi b_\alpha \ , \\ \hat{A}_{\alpha\beta\gamma 3} &= A_{\alpha\beta\gamma 3} + 3b_{[\alpha}C_{\beta\gamma]} + 6\partial_{[\alpha}V B_{\beta\gamma]} \ , \end{aligned} \tag{18}$$

for $\alpha, \beta, \ldots = 0, 1, 2$ and

$$b_3 = \lambda_3 = 0 \ . \tag{19}$$

In this form, the action (17) can be interpreted as the Nambu-Goto action for a fundamental 3-brane. The term involving $\hat{A}$ is the Wess-Zumino term.

It is clear from the explicit expressions of b_α and λ_α, that the target space of the fundamental 3-brane is twelve dimensional, with U and V giving rise to two extra target space coordinates. It is evident that both of these coordinates are spacelike and, thus, the signature of the target space that emerges here naturally is $(11, 1)$(see section 3). Moreover, the target space theory obtained from the Nambu-Goto action in (17) can be interpreted as the F-theory.

Note that because of the double dimensional reduction procedure we adopted here, the Nambu-Goto action for the 3-brane in (17) appears in a gauge given by (19). Hence, it does not have the full twelve dimensional general coordinate invariance. However, this is likely to be a generic phenomenon. That is, two such conditions are always necessary, if a Nambu-Goto action were to be derivable from the Born-Infeld action of type IIB Dirichlet 3-brane. This can be seen by a simple counting as follows.

The D-brane action of the type II strings already has the metric $g_{\alpha\beta}$ induced by ten target space coordinates, see (1). Now, if the target space has two extra coordinates, as expected for a fundamental 3-brane, then there should be 8 (= number of extra target space coordinates $\times$ the world volume dimension) extra degrees of freedom, corresponding to $\partial_\alpha X^{10}$ and $\partial_\alpha X^{11}$, $\alpha = 0, 1, 2, 3$. The type IIB D-3-brane does have extra degrees of freedom on the world volume, coming from $\mathcal{F}_{\alpha\beta}$, but they are 6 only in number, falling short by 2! This is the origin of the two conditions in (19).

In the case of 2-branes, the target space is eleven dimensional, and the world volume is three dimensional, so the required number of extra degrees of freedom is 3. The type IIA D-2-brane has extra degrees of freedom on the world volume, coming from $\mathcal{F}_{mn}$, which are also 3 in number, exactly what is needed. Then, our counting argument suggests that a Nambu-Goto action for a fundamental 2-brane should be derivable from the type IIA Dirichlet 2-brane action, without any gauge condition. This is indeed the case, as shown in [27, 15].

Perhaps, instead of the type IIB Dirichlet 3-brane action, if a more general or an altogether different action with more degrees of freedom is used, then may be a Nambu-Goto action for a fundamental 3-brane is derivable without any gauge condition. However, we will not pursue it here (see discussion in section 5).

That such a phenomenon generically occurs, and that the resulting target space action does not have the full twelve dimensional general coordinate invariance, may actually be a boon in disguise. As mentioned in [15], this may perhaps be related to the way F-theory implements supersymmetry, circumventing Nahm's theorem regarding realisations

of supersymmetry in spacetime with dimensions > 11 [30].

3 Target space metric

The twelve dimensional target space metric of the F-theory is easy to read off from (17). Before doing so, let us now restore the ϕ-dependence. We only need to use equations (6) and (12). Then, as can be seen easily, b_α and λ_α given in (18) are to be replaced as follows:

$$b_\alpha \to e^{-\frac{\phi}{2}} b_\alpha \ , \quad \lambda_\alpha \to e^{\frac{\phi}{2}} \lambda_\alpha \ . \tag{20}$$

The action (17), after these replacements, can be written as

$$S = \int d^4x \left(\sqrt{-\det(\hat{g}_{\alpha\beta})} + \frac{\epsilon^{\alpha\beta\gamma\delta}}{6} \hat{A}_{\alpha\beta\gamma\delta} \right) \ , \tag{21}$$

where

$$\hat{g}_{\alpha\beta} = e^{-\frac{\phi}{2}} g_{\alpha\beta} + e^{-\phi} b_\alpha b_\beta + e^{\phi} \lambda_\alpha \lambda_\beta \ . \tag{22}$$

The action in (21) now has the correct ϕ-dependence.

We now rewrite $\hat{g}_{\alpha\beta}$, in a form which makes explicit the origin of type IIB duality symmetries. Let us define a two component vector $\mathbf{Z}_\alpha$ and a matrix $\mathcal{M}$ as follows:

$$\mathbf{Z}_\alpha = \begin{pmatrix} \partial_\alpha V - C_\alpha \\ \partial_\alpha U + B_\alpha \end{pmatrix} \ , \quad \mathcal{M} = \frac{1}{\lambda_2} \begin{pmatrix} 1 & \lambda_1 \\ \lambda_1 & |\lambda|^2 \end{pmatrix} \ , \tag{23}$$

where $\lambda = \lambda_1 + i\lambda_2 \equiv \chi + ie^{-\phi}$. Then

$$\hat{g}_{\alpha\beta} = e^{-\frac{\phi}{2}} g_{\alpha\beta} + {\mathbf{Z}_\alpha}^T \mathcal{M} \mathbf{Z}_\beta \ . \tag{24}$$

Similarly, let us now define

$$\begin{aligned} dU^1 &= dV - \frac{1}{2}(C_{\mu 9} - C_{9\mu}) dX^\mu \ , \\ dU^2 &= dU + \frac{1}{2}(B_{\mu 9} - B_{9\mu}) dX^\mu \ . \end{aligned} \tag{25}$$

Note that these equations for U^1 and U^2 can be solved locally and, depending on the topological properties of $B_{\mu\nu}$ and $C_{\mu\nu}$, may also be solved globally. The twelve dimensional target space line element is then given by

$$ds^2 = e^{-\phi/2} g_{\mu\nu} dX^\mu dX^\nu + e^\phi (dU^1)^2 + 2e^\phi \chi dU^1 dU^2 + (e^{-\phi} + e^\phi \chi^2)(dU^2)^2 \ . \tag{26}$$

In this form, it is clear that U^1 and U^2 are the two extra target space coordinates, and that both of them are spacelike. Furthermore, as can be seen easily, the metric on this internal two dimensional space is given by

$$g_{\rm int} = \mathcal{M} \ , \tag{27}$$

where $\mathcal{M}$ is defined in (23). This metric describes a torus, whose two radii of are proportional to $e^{-\phi}$ and e^{ϕ}. Thus, locally, the twelve dimensional target space of F-theory is a product of a ten dimensional spacetime and a torus, with a fixed Kähler class (see also section 4). That is, the twelve dimensional space is an elliptic fibration over a ten dimensional base [13]. The duality properties of this torus and its relation to the type IIB string duality symmetries will be discussed now.

4 Geometric Interpretation of $SL(2, Z)$ symmetry

Let us first briefly mention how $SL(2, Z)$ symmetry appears in string theory. Evidence of $SL(2, Z)$ symmetry (S-duality) was first seen in the toroidal compactification of heterotic string to four dimensions. This is a symmetry of heterotic string equations of motion [2, 3]. It is manifest in Einstein frame, and a subgroup of these transformations inverts the string coupling constant and simultaneously interchanges electric and magnetic fields.

Type II string compactified on $K_3 \times T^2$ also exhibits this symmetry as a strong-weak coupling duality in four dimensions [4]. The geometric realisation of this symmetry in both heterotic case as well as type II case was achieved through the string-string duality conjecture in six dimensions, between heterotic string on T^4 and type IIA string on K_3. On toroidal compactification, both of them have $SL(2, Z)$ as the S-duality symmetry which can now be interpreted as modular group of internal torus in the following manner. Torus compactification of both of these six dimensional theories to four dimensions gives $O(2, 2, Z)$ T-duality group. $O(2, 2, Z)$ is isomorphic to $SL(2, Z) \times SL(2, Z)$, which act on the complex structure modulus and the Kähler modulus of a torus respectively. It was shown in [32] that for a fixed Kähler modulus the S-duality group of heterotic string is the modular group ($SL(2, Z)$ corresponding to the complex structure deformation of torus) of the torus on which type IIA string is compactified. Similarly the S-duality group of type IIA string is modular group of the torus on which heterotic string is compactified. Six dimensional string-string duality thus helps us establish the cross connection between S-duality of one string theory with the T-duality of another allowing us to give geometric interpretation of S-duality symmetry of four dimensional string theory.

Another string theory which has $SL(2,Z)$ S-duality symmetry is type IIB string theory in ten dimensions. This symmetry mixes the fields in the NS and RR sector and leaves the equations of motion invariant. Under $SL(2,Z)$ transformations, the Einstein frame metric (see below) and the self-dual four form field are invariant; and,

$$\lambda \rightarrow \frac{a\lambda+b}{c\lambda+d}\ , \qquad \begin{pmatrix} C_{\alpha\beta} \\ B_{\alpha\beta} \end{pmatrix} \rightarrow (\Omega^T)^{-1} \begin{pmatrix} C_{\alpha\beta} \\ B_{\alpha\beta} \end{pmatrix}\ , \tag{28}$$

where $\lambda = \lambda_1 + i\lambda_2 \equiv \chi + ie^{-\phi}$ and

$$\Omega = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,Z)\ .$$

The above transformation of λ can also be written as $\mathcal{M} \rightarrow \Omega\mathcal{M}\Omega^T$, where and $\mathcal{M}$ is defined in equation (23). This $SL(2,Z)$ symmetry of the ten dimensional type IIB strings had no natural geometric interpretation as a modular group of internal torus. In nine dimensions, however, it can be related to two torus compactification of M-theory [6].

In the previous section we saw that the D-3-brane action can be interpreted as the Nambu-Goto action for a fundamental 3-brane whose target space is twelve dimensional with (11, 1) signature. The internal two dimensional metric is covariant under $SL(2,Z)$ transformations. As mentioned earlier $SL(2,Z)$ symmetry of type IIB string is manifest in the Einstein frame. We will therefore first absorb the dilaton factor and write the ten dimensional metric in the Einstein frame. The ten dimensional Einstein frame metric is given by

$$g^E_{\mu\nu} = e^{-\phi/2} g_{\mu\nu}. \tag{29}$$

Substituting this in (24) we see that the Nambu-Goto action is manifestly invariant under $SL(2,Z)$ transformations (28) if $\mathbf{Z}_\alpha \rightarrow (\Omega^T)^{-1}\mathbf{Z}_\alpha$. In the target space, the metric on the internal torus $g_{int} = \mathcal{M}$ (see (27)) transforms under $SL(2,Z)$ as

$$g_{int} \rightarrow \Omega g_{int} \Omega^T\ , \qquad \Omega \in SL(2,Z) \tag{30}$$

and the toroidal coordinates U^1 and U^2 as well as the two forms $B_{\mu\nu}$ and $C_{\mu\nu}$ transform as,

$$\begin{pmatrix} U^1 \\ U^2 \end{pmatrix} \rightarrow (\Omega^T)^{-1} \begin{pmatrix} U^1 \\ U^2 \end{pmatrix}\ ,\ \begin{pmatrix} C_{\mu\nu} \\ B_{\mu\nu} \end{pmatrix} \rightarrow (\Omega^T)^{-1} \begin{pmatrix} C_{\mu\nu} \\ B_{\mu\nu} \end{pmatrix}\ . \tag{31}$$

We thus see that $SL(2,Z)$ symmetry of type IIB string theory in ten dimensions has a direct geometric interpretation in terms of the modular group of the internal torus. The $SL(2,Z)$ modulus of type IIB string theory in ten dimension is the complex structure modulus of this torus. Definition of torus coordinates U^1 and U^2, given in (25), involves

B and C fields and, therefore, the twelve dimensional space can be interpreted as an elliptic fibration. If we set both B and C to zero, then $U^1 = V$ and $U^2 = U$. In this case the twelve dimensional space is a direct product of ten dimensional spacetime and the internal torus. The fact that this toroidal compactification does not involve second $SL(2, Z)$ alluded to earlier in this section implies that this torus has a fixed Kähler class. This also fits in nicely with the construction of 24 seven branes, and their relation to compactification of F-theory on elliptically fibered K_3 manifolds, given by Vafa [13]. By showing the relation of $SL(2, Z)$ with modular group of the internal torus we have put this symmetry of type IIB strings on the same footing as the $SL(2, Z)$ S-duality symmetry of four dimensional $N = 4$ supersymmetric string theory.

We would like to note here an interesting implication of this interpretation of $SL(2, Z)$ symmetry. Strong-weak coupling duality transformation in type IIB string theory which is a subgroup of the $SL(2, Z)$ duality, exchanges the torus coordinates U^1 and U^2. As seen in section 3, the two radii of the torus are proportional to $e^{-\phi}$ and e^{ϕ}, where $e^{-\phi}$ is the string coupling. We thus see that both in strong coupling or weak coupling limit, this theory appears to be eleven dimensional theory with $(10, 1)$ signature. Perhaps, this is related to the conjectured self-duality of M-theory [29].

5 Discussion

To summarise, we start from the type IIB Dirichlet 3-brane action. Performing first a double dimensional reduction, and then applying the methods of [27, 15], we obtain a Nambu-Goto action. It is interpreted as the world volume action of a fundamental 3-brane, and its target space theory as F-theory. We find that the target space is twelve dimensional, with signature $(11, 1)$. Locally, it is a product of a ten dimensional spacetime and a torus, with a fixed Kähler class. That is, the twelve dimensional space is an elliptic fibration over a ten dimensional base space [13].

Also, the $SL(2, Z)$ symmetry of type IIB string has an explicit geometric interpretation: the $SL(2, Z)$ modulus of type IIB string is the complex structure modulus of the torus fiber. Moreover, the two radii of the torus are proportional to $e^{-\phi}$ and e^{ϕ}, where $e^{-\phi}$ is the string coupling. Hence, both in the strong and weak coupling limit, the twelve dimensional theory appears to be eleven dimensional with (10,1) signature, which is perhaps related to the conjectured self-duality of M-theory.

However, because of the double dimensional reduction, the Nambu-Goto action ap-

pears in a gauge and, hence, it does not have the full twelve dimensional general coordinate invariance. We also showed, by a simple counting argument, that this is likely to be a generic phenomenon. This may actually be a boon in disguise, as it may be related to the way F-theory implements supersymmetry, circumventing Nahm's theorem regarding realisations of supersymmetry in spacetime with dimensions > 11.

We conclude by pointing out a couple of aspects related to F-theory, whose understanding is likely to provide deeper insights. One needs to understand thoroughly the gauge fixing we encountered in section 2. For this purpose, perhaps, one may start from an action more general than, or altogether different from, the Born-Infeld action for type II D-3-branes, and which has enough number of fields to provide more degress of freedom. One natural place to start may be the following: The duality symmetry of the Born-Infeld action is manifest only at the level of equations of motion [33]. Writing this action in a manifestly duality invariant way, say, in analogy with the Maxwell's case treated in [2], will naturally introduce new fields. Such an action, if found, may lead to the Nambu-Goto action for a fundamental 3-brane, without any gauge conditions.

Note that the target spacetimes of fundamental string, 2-brane, and 3-brane are, respectively, 10, 11, and 12 dimensional. In all these cases, the dual of the fundamental branes is always a 5-brane. The non trivial role of 3-branes and 5-branes in F-theory is also seen in [34] from current algebra considerations. These observations suggest that perhaps 5-brane is the really fundamental object to study. In such an event, its study is likely to reveal important facts common to all these theories. A similar point of view is being advocated in [29] also.

Note Added: F-theory is expected to lead to M-theory upon double dimensional reduction. Hence, our double dimensional reduction procedure, adopted here for the sake of clarity and simplicity, may appear to imply that one is dealing with 2-brane and M-theory, and not with 3-brane and F-theory.

However, this is not the case. The two extra coordinates U and V that emerge in our approach are distinct from any of the original ten coordinates X^μ, leading thus to the twelve dimensional F-theory. Also, only for the four dimensional 3-brane action (3) do the string coupling factors e^ϕ appear with the coefficients as given in (5) and (6). As seen in sections 3 and 4, it is these coefficients which eventually lead to the correct geometric interpretation of type IIB $SL(2,Z)$ symmetry.

If one were dealing with 2-brane action then the $e^{-\frac{\phi}{2}}$ factor in equation (5) would be

replaced by $e^{-\frac{2\phi}{3}}$. The e^{ϕ} factors in (6) would also be replaced correspondingly. Then, the resultant Nambu-Goto action would not lead to a direct geometric interpretation of type IIB $SL(2,Z)$ symmetry.

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